Small Dimension PDE for Discrete Asian Options

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Abstract

This paper presents an efficient method for pricing discrete Asian options in presence of smile and non proportional dividends. Using an homogeneity property, we show how to reduce an n0 dimensional problem to a 1 or 2 dimensional one. We examine different numerical specifications of our dimension reduced PDE using a Crank Nicholson method (interpolation method, grid boundaries, time and space steps) as well as the extension to the case of non proportional discrete dividends, using a jump condition. We benchmark our results with Quasi Monte-Carlo simulation and a multi-dimensional PDE.

Key words: Discrete Asian Option, Homogeneity, PDEs, Crank-Nicholson, Non proportional Dividends, Smile
JEL Classification: G12, G13
1991 MSC: classification: 60H35, 62P05, 65N06

\* We would like to thank Nicole El Karoui for interesting remarks and two anonymous referees. The ideas expressed herein are the author’s and do not necessarily reflect those of Goldman Sachs or BNP-Paribas

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1 Introduction

Asian options are securities with a payoff depending on the average value of an underlying stock, index, interest rates over some time period. First introduced in Tokyo\(^3\), Asian Options are among the most popular path-dependent options, since their characteristics capture, in a way, the whole trajectory of the underlying, with a reduced exposure to volatility in most cases. The common belief that these options should be cheaper than their corresponding string of vanilla options is not strictly accurate. However, it happens to often be the case in various practical cases (see Geman and Yor (1993)) for a discussion). In addition, Asian options are less sensitive to possible spot manipulations or extreme movements at settlement and offer much flexibility in the way the average is settled. From a trader’s point of view, the delta of an Asian option naturally decreases since part of the average becomes known after an observation date. The hedging strategy is therefore eased, compared with regular options. Consequently, Asian options have become very attractive for investors since they provide a customized cheap way to hedge periodic cash-flows (see Longstaff (1995) for a discussion of the efficiency of Asian interest-rate options for corporations with reasonably predictable cash flows). Nonetheless, such options have turned out to be much more difficult to value than standard options.

Previous research was intensively focused on continuous time Asian options using Black-Scholes (1973) assumptions. However, traded Asian options are based on a discrete time sampling and the underlying security can exhibit a pronounced volatility smile as well as non-proportional dividends.

The existing very extensive literature has at least two major drawbacks. Previous works attempting at approximate closed forms solutions fail to adapt to more complex volatility models, like the Dupire (1993a), (1993b) and Deman and Kani (1994) ones, as well as to American type features. This includes the work of Vorst (1996), (1992) (approximation of the arithmetic average with a modified geometric one), Geman and Yor (1993) (closed formula for the Laplace transform of the option, by means of Bessel processes), Turnbull and Wakeman (1991) (use of a lognormal density to approximate the sum of lognormal density), Levy (1992) and Jacques (1996) (use of an Edgeworth expansion to match higher moments), Zhang (1996) (derivation of an approximation using the geometric Asian option, by means of a Taylor expansion) and Milevsky and Posner (1997). (use of the asymptotic limit of the sum of lognormal density known as the reciprocal gamma density).

Similarly, previous works on numerical methods do not account for volatility

\(^3\) hence the name of Asian options as opposed to American, European or Bermudean ones.

The motivation of this paper is to provide an efficient method for pricing discrete Asian options with a deterministic volatility as specified in Dupire (1993a), (1993b) and Deman and Kani (1994) as well as non-proportional discrete dividends. These two features are far more realistic than Black Scholes assumptions for equity derivatives pricing. Using an homogeneity property, which can be read in the same lines as the change of variables of Rogers and Shi (1995), we show how to obtain a parabolic partial differential equation with diminished spatial variables. We reduce an $n$ dimensional problem to a 1 or 2 dimensional one, where $n$ stands for the number of fixings. This is of considerable interest for the efficient computation of discrete Asian options. This generalizes to discrete Asian options the dimension reduction technique found for continuous Asian options by Rogers and Shi (1995). We show that the homogeneity property is coarsely conserved within a deterministic volatility structure, consistent with the smile as in the Dupire model. This is also true for non proportional discrete dividends, solved with a jump condition. We can still infer call prices by means of the homogeneity property. We derive a PDE for the computation of the Asian option and solve it with the standard Crank Nicolson method. Because of the dimension reduction, we need to interpolate our conditional price at each fixing dates. The rest of the article tackles the issue of numerical specifications for the finite difference method (grid boundaries, time and space steps). We compare our result with a Quasi Monte Carlo simulation based on Sobol sequences.

The remainder of this paper is organized as follows. In section 2, we present the standard result on PDEs for Asian options and discuss how to reduce the dimension. In section 3, we explain how to reduce the dimension of the problem using an homogeneity property and a conditional expectation method. We introduce a modified strike variable. This leads to a 2 dimensional PDE which in the case of the Black Scholes diffusion is only 1 dimensional. In section 4, we show how to account for non homogeneous situation either implied by the smile effect or by discrete non proportional dividends. Section 5 studies the numerical part of the results. It compares our method with a benchmark price given by a Sobol Quasi Monte Carlo simulation. It also explains how to choose the mesh and examines the efficiency of the vega and dividend correction. We conclude briefly in section 6 giving some further developments.
2 How to reduce the Dimension?

2.1 Mathematical Framework

We consider a continuous time trading economy with an infinite horizon. The uncertainty is characterized by a complete probability space \((\Omega, \mathcal{F}, Q)\) where \(\Omega\) is the state space, \(\mathcal{F}\) is the \(\sigma\)-algebra representing the measurable events, and \(Q\) is the risk neutral probability measure, assumed to be unique in a complete market with no arbitrage opportunity. The information evolves according to the augmented right continuous complete filtration \(\{\mathcal{F}_t, t \in \mathbb{R}^+\}\) generated by a standard one dimensional Brownian Motion \(\{W_t, t \in \mathbb{R}^+\}\). We assume the evolution of the underlying price process \((S_t)_{t \in \mathbb{R}^+}\) is described by a Stochastic Differential Equation (1)

\[
dS_t = r_t S_t dt + S_t \sigma (t, S_t) dW_t
\]

with an initial condition \(S_0 = x\), \(r_t\) is the deterministic risk free interest rate and \(\sigma (t, S_t)\) is either constant (Black Scholes model) or locally deterministic (like in the Dupire and CEV models, in fact is a deterministic function of \(S_t\)).

The solution for the underlying process is given by

\[
S_t = xe^{\int_0^t (r_u - \frac{1}{2} \sigma^2(u, S_u)) du + \int_0^t \sigma(u, S_u) dW_u}
\]

for \(0 \leq v \leq t\). It is worth noticing that the underlying process is not perfectly homogeneous (of degree 1) with respect to \(x\), as soon as the volatility structure \(\sigma (t, S_t)\) depends on \(S_t\). We denote by \(\mu\) a density measure over the interval \([0, T]\) with a continuous density \(\rho_t\) and some atoms \(\sum_{i=1}^n \alpha_i \delta_{t_i}\) at points \((t_i)_{i=1,n}\) representing some fixing dates with \(t_n = T\), and \((\alpha_i)_{i=1,n}\) some weights, either positive or negative. The averaging measure \(\mu\) is not necessary absolutely continuous with respect to the Lebesgue measure and is not necessary of total measure 1. This enables us to be very general, allowing for discrete or continuous-time averaging, fixed or floating strike as explained in exhibit 1. We focus at the following option payoff:

\[
(A_T - K)^+
\]

where \(K\) is a real number. The running average given by:

\[
A_t = \int_0^t S_u \mu (du)
\]
<table>
<thead>
<tr>
<th>Asian option type</th>
<th>Discrete underlying</th>
<th>Continous underlying</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed strike</td>
<td>$\left(\frac{\sum_{i=1}^{n} S_t}{n} - K\right)^+$</td>
<td>$\left(\frac{\int_0^T S_t dt}{T} - K\right)^+$</td>
</tr>
<tr>
<td></td>
<td>$\mu(dt) = \sum_{i=1}^{n} \delta_i(t)$</td>
<td>$\mu(dt) = \frac{1}{T} \int_0^{T} \delta_i(t) dt$</td>
</tr>
<tr>
<td>Floating strike</td>
<td>$\left(\frac{\sum_{i=1}^{n} S_t}{n} - S_T\right)^+$</td>
<td>$\left(\frac{\int_0^T S_t dt}{T} - S_T\right)^+$</td>
</tr>
<tr>
<td></td>
<td>$\mu(dt) = \sum_{i=1}^{n} \delta_i(t)$ - $\delta_T(t)$</td>
<td>$\mu(dt) = \frac{1}{T} \int_0^{T} \delta_i(t) dt - \delta_T(t)$</td>
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**Exhibit 1:** Payoff and measure for different type of Asian option the sign $\delta_u(.)$ denotes the Dirac function at the point $u \in \mathbb{R}$

### 2.2 Determination of a small dimension PDE

A brute force PDE for a discrete Asian option with $n$ fixing dates would consist in a $n + 1$ dimensional PDE. The option depends on $n$ variables $S_{t_1}, ..., S_{t_n}$ and the time $t$. This becomes soon intractable because of the high dimension of the problem. The complexity of a finite difference method increases exponentially with respect to the dimension. To reduce the dimension, we suggest two strategies: first, we extend the method of Rogers and Shi (1995), derived for the Black Scholes case, to non-constant volatility structures as imposed by the smile effect. Indeed, we see that this method is only appropriate for homogeneous diffusions. However, for a diffusion implied by the Dupire method, this is inappropriate. A second approach, which can offer a solution to this particular case, exploits the homogeneity property of the option underlying and price.

#### 2.2.1 Traditional Black and Scholes PDEs

Before embarking into some dimension reduction consideration, we show how to adapt traditional PDEs for the Asian option to non-constant volatility structures. The standard PDE derived for continuous-time Asian options (as explained in Ingersoll (1987) or Forsyth, Vetzal and Zvan (1998)) leads to the following expression in the case of non constant volatility structure:

$$C_t + \frac{1}{2} \sigma^2 (t, S_t) S_t^2 C_{ss} + r S C_s + s C_t - r C = 0$$

where $C_t$ respectively $C_s, C_I, C_{ss}$ denotes the first order partial derivative function with respect to the time, respectively the underlying, the running sum
and the second order partial derivative function with respect to the underlying. \( I \), the running sum is defined as \( \int_0^t S_u du \) in the continuous case and by \( \sum_{i=1}^n S_t \mathbf{1}_{t_i \leq t} \) in the discrete one. When deriving the PDE with respect to the running average denoted by \( A \), we find the following PDE (see for instance Barraquand and Pudet (1996) or Wilmott (1993))

\[
C_t + \frac{1}{2} \sigma^2 (t, S_t) S_t^2 C_{ss} + r S C_s + \frac{1}{T} (S_t - A_t) C_A - r C = 0
\]

The difference with the standard Black Scholes PDE comes from the dependence in the underlying of the volatility structure. For discrete Asian options, these equations transform to the same one dimensional PDE:

\[
C_t + \frac{1}{2} \sigma^2 (t, S_t) S_t^2 C_{ss} + r S C_s - r C = 0
\]

with the condition at the observation date

\[
C \left( t_i^-, S, A_t^- \right) = C \left( t_i^+, S, A_t^- + \alpha_i S_t \right)
\]

where \( C (t, S, A_t) \) denotes the call value at time \( t \) with underlying \( S \) and average \( A_t \). For \( n \) fixing dates, this represents a set of \( n \) one-dimensional PDEs and is computationally time-consuming.

### 2.2.2 Change of variable

As suggested by Rogers and Shi (1995) in the case of the Black Scholes model, we can use a change of variable to reduce the dimension. Extending their works, we show that the PDE to be satisfied by the option price is a two dimensional one. The price of a call option is expressed as the expected value of the discounted pay-off under the risk neutral probability measure with a risk free rate assumed to be deterministic:

\[
C_t = \mathbb{E}_Q \left[ e^{-\int_t^T r_s ds} \left( \int_0^T S_s \mu(ds) - K \right)^+ | \mathcal{F}_t \right]
\]

We define

\[
f (t, k, S_t) = \mathbb{E}_Q \left[ \left( \int_t^T \frac{S_s}{S_t} \mu(ds) - k \right)^+ | \mathcal{F}_t \right]
\]

The option price is given by
By Itô’s formula, we notice that the function $f$ has enough smoothness to apply Itô’s formula to the equation (5), we get

$$C_t = e^{-\int_{t}^{T} r_s ds} \mathbb{E}_Q \left[ \left( \int_{0}^{T} S_s \mu (ds) - K \right)^{+} | \mathcal{F}_t \right]$$

$$= e^{-\int_{t}^{T} r_s ds} S_t \mathbb{E}_Q \left[ \left( \int_{t}^{T} \frac{S_s \mu (ds) - K - f_s}{S_t} \right)^{+} | \mathcal{F}_t \right]$$

$$= e^{-\int_{t}^{T} r_s ds} S_t f \left( t, \frac{K - f_s}{S_t}, S_t \right)$$

$$= e^{-\int_{t}^{T} r_s ds} S_t f \left( t, Y_t, S_t \right)$$

(5)

where

$$Y_t = \frac{K - f_s}{S_t}$$

By Itô’s formula,

$$dY_t = \left( -\mu (dt) - r_t Y_t + \sigma^2 (t, S_t) Y_t \right) dt - Y_t \sigma (t, S_t) dW_t$$

Since $e^{-\int_{t}^{T} r_s ds} C_t$ is a martingale, its deterministic part should be equal to zero. We notice that the function $f : t, k, s \mapsto f (t, k, s)$ is jointly continuous in $t, k$ and $s$, decreasing in $t$ and decreasing convex in $k$. Assuming that the function $f$ has enough smoothness to apply Itô’s formula to the equation (5), we get

$$d \left( e^{-\int_{t}^{T} r_s ds} C_t \right) = S_t \left( f_t (t, Y_t, S_t) dt + f_k (t, Y_t, S_t) dY_t + \frac{1}{2} f_{kk} (t, Y_t, S_t) \langle dY_t \rangle + f_s (t, Y_t, S_t) dS_t + \frac{1}{2} f_{ss} (t, Y_t, S_t) \langle dS_t \rangle + f_k (t, S_t) dS_t + \langle dS_t, df (t, Y_t, S_t) \rangle \right)$$

the deterministic term should be equal to zero, leading to

$$0 = S_t \left( f_t + r_t f + \frac{1}{2} (Y_t \sigma (t, S_t))^2 f_{kk} - r_t Y_t f_k \right) dt - f_k \mu (dt) + \left( r_t f_s + \frac{1}{2} \sigma^2 (t, S_t) S_t f_{ss} - Y_t \sigma^2 (t, S_t) f_{ks} \right) dt$$

(6)

If the measure $\mu$ has a continuous density $\rho_t$ and some atoms $\sum_{i=1}^{n} \alpha_i \delta_{t_i}$, and if we denote by $g (t, y, s) = e^{-\int_{t}^{T} r_s ds} f (t, y, s)$ the equation (6) can be rewritten as the following PDE

$$\frac{\partial}{\partial t} g + A g = 0$$

(7)

with

$$A = \begin{pmatrix} \frac{1}{2} y^2 \sigma^2 (t, s) \frac{\partial^2}{\partial y^2} - (\rho_t + \sum_{i=1}^{n} \alpha_i \delta_{t_i} + r_t y) \frac{\partial}{\partial y} \\ + \frac{1}{2} \sigma^2 (t, s) s \frac{\partial^2}{\partial s^2} + r_t \frac{\partial}{\partial s} - y \sigma^2 (t, s) \frac{\partial}{\partial y} \end{pmatrix}.$$
At a point of an atom \( t_i \), wit value \( \alpha_i \), we get by an arbitrage argument (the value of \( g \) has to be continuous)

\[
g(t_i^-, y) = g(t_i^+, y + \alpha_i)
\]

With the change of variable, the boundary condition is equal to

\[
g(T, y, s) = (\alpha_T - y)^+\quad (8)
\]

The call price is obtained by:

\[
C_{t=0} = xg(0, \frac{K}{x}, x)
\]

The variable \( Y_t \) can be interpreted as a conditional strike as shown in the next subsection.

2.2.3 Homogeneous case

For an homogeneous underlying (like the Black Scholes model), the function \( f(t, k, S_t) \) does not depend on the underlying price \( S_t \). It reduces to

\[
f(t, k) = E_Q \left[ \left( \int_t^T S_t \mu(ds) - k \right)^+ | S_t = 1 \right]
\]

as shown in Rogers and Shi (1995) for instance. This property can also be applied to stochastic volatility models (like in Hull and White (1987), Wiggings (1987), Melino and Turnbull (1990), Stein and Stein (1991), Amin and Ng (1993) and Heston (citeyearhe:cf)). This comes from the fact that stochastic volatility models are particularly tractable when using conditional expectation.

Furthermore, for the Black Scholes model, the volatility structure is a constant. The PDE (7) simplifies into a one dimensional one:

\[
\frac{\partial}{\partial t} g + \tilde{A} g = 0
\]

with the diffusion operator given by:

\[
\tilde{A} = \frac{1}{2} y^2 \sigma^2 \frac{\partial^2}{\partial y^2} - \left( \rho_t + \sum_{i=1}^n \alpha_i \delta_{t_i} + r_t y \right) \frac{\partial}{\partial y}
\]

and still the same boundary conditions (4). The result has interesting implications for the efficiency of numerical schemes. For the two types of options,

\[
4 \quad g(T, y, s) = \left( \frac{\int_0^T S_{t \mu(ds) + \alpha T S_T - K}}{S_{s_n}} \right)^+ = (\alpha_T - y)^+
\]
fixed and floating strike ones, the PDE is only one dimensional. However, this property is not easily adaptable to more complex volatility structure.

One of the important but often disregarded property of a geometric Brownian motion is its homogeneity property. This is an appropriate method for the Asian option when looked at as a conditional expectation calculation. The price of the discrete Asian option can be rewritten as the following conditional expectation:

\[
C = E_Q \left[ E_{Q} \left[ \left( e^{-\int_{0}^{T} r_s ds} \left( \sum_{i=1}^{n} \alpha_i S_t - K \right)^+ \right) \left| S_{t_1}, ..., S_{t_{n-1}} \right. \right] \right] \tag{11}
\]

Such a conditional expectation can be interpreted as a call option with a strike equal to \( (K - \sum_{i=1}^{n-1} \alpha_i S_t) \). The homogeneity (of degree one) of the call option price leads to the following remark. Denoting by \( C(x,k) \) the call price with an initial underlying level of \( x \) and a strike of \( k \), we have

\[
C(x,k) = C\left(\frac{xk}{k_0}, k_0 \right) \frac{k}{k_0} \tag{12}
\]

The knowledge of call prices for one strike but different underlying levels is consequently equivalent to the one for all pair of strikes and underlying levels. The finite difference method provides call prices for different values of the underlying. Using an interpolation, we can infer a continuum of prices of the call option for different underlying value. This implies the knowledge of any call price.

The algorithm works as follows: it implies to calculate the call price between two fixing dates for a given level of strike. This is done by a Crank Nicolson method with a backward propagation. When the previous fixing date is reached, we infer call prices for different levels of strike by means of the homogeneity property using the equation (12). The homogeneity property works at any date. However, at the fixing date, one must be sure to interpolate using call prices which includes the fixing. After the interpolation by means of the homogeneity property, we continue the backward propagation.

3 PDE solving for the Homogeneous case

We concentrate on discrete Asian options, often of more interest. The equation in the case of the Black Scholes model is either the one dimensional one as explained on the preceding section or the simple Black Scholes equation, with at each fixing dates, the use of the homogeneity property.
3.1 Discretisation of the PDE: Crank Nicolson Method

We use a Crank Nicolson finite difference method. The straightforward discretisation of the PDE derived for the Asian option provides a spurious solution. Indeed, it is more appropriate to use the logarithmic change of variable, as argued by Brennan and Schwartz (1978), Hull and White (1990). In the latter case, the diffusion operator is uniformly elliptic. We denote by $C_{i,j}$ the discretised function where the first variable $i$ stands for the time, whereas the second one $j$ for the space variable. We get the following discretised scheme:

$$
\frac{C_{i+1,j} - C_{i,j}}{\Delta T} + \left( r - \frac{\sigma^2}{2} \right) \left( \frac{C_{i+1,j+1} - C_{i+1,j-1}}{4\Delta S} + \frac{C_{i,j+1} - C_{i,j-1}}{4\Delta S} \right)
+ \frac{\sigma^2}{2} \left( \frac{C_{i+1,j+1} - 2C_{i+1,j} + C_{i+1,j-1}}{\Delta S^2} + \frac{C_{i,j+1} - 2C_{i,j} + C_{i,j-1}}{\Delta S^2} \right)
= r \left( \frac{C_{i+1,j} + C_{i,j}}{2} \right)
$$

or after grouping the terms

$$a_{i,j}C_{i+1,j+1} + a_{i,j}C_{i,j} + a_{i,j-1}C_{i,j-1}
= a_{i+1,j+1}C_{i+1,j+1} + a_{i+1,j}C_{i+1,j} + a_{i+1,j-1}C_{i+1,j-1}
$$

with

$$a_{i+1,j+1} = \left( r - \frac{\sigma^2}{2} \right) \frac{1}{4\Delta S} + \frac{\sigma^2}{2} \frac{1}{2\Delta S^2} \quad a_{i+1,j} = \frac{1}{\Delta T} - \frac{\sigma^2}{2} \frac{2}{2\Delta S^2} - \frac{r}{2}
$$
$$a_{i+1,j-1} = - \left( r - \frac{\sigma^2}{2} \right) \frac{1}{4\Delta S} + \frac{\sigma^2}{2} \frac{1}{2\Delta S^2} \quad a_{i,j+1} = - \left( r - \frac{\sigma^2}{2} \right) \frac{1}{4\Delta S} - \frac{\sigma^2}{2} \frac{1}{2\Delta S^2}
$$
$$a_{i,j} = \frac{1}{\Delta T} + \frac{\sigma^2}{2} \frac{2}{2\Delta S^2} + \frac{r}{2} \quad a_{i,j-1} = \left( r - \frac{\sigma^2}{2} \right) \frac{1}{4\Delta S} + \frac{\sigma^2}{2} \frac{2}{2\Delta S^2}
$$

This is solved by a standard LU method as explained in Press et al. (1992).

3.2 Interpolation and Extrapolation at observation dates

The actual PDE which is discretized is the one of Black Scholes in the homogeneous case and the modified one with local volatility as in Dupire (1993b). In order to illustrate this methodology, let’s take the example of a very simple Asian option, that is a fixed strike, two fixings average call. We want to evaluate

$$C_0 = \mathbb{E}_Q \left[ e^{-r(t_2-t_1)} \left( \frac{S_{t_1} + S_{t_2}}{2} - K \right)_{+} | \mathcal{F}_0 \right]$$

10
We use the Crank-Nicolson algorithm. At a given point in the middle of the grid and referred to as by its time \( t_i \) and its status \( S_{t_j} \), the grid will exhibit the values of

\[
C_{i,j} (K) = C_i (S_{t_j}, K) = \mathbb{E}_Q \left[ e^{-r(t_2-t_i)} \left( \frac{S_{t_j} + S_{t_2}}{2} - K \right)^+ | S_{t_i} = S_{t_j} \right]
\]

i.e. half the price of a call of strike \( K \), for \( t_i \in [t_1, t_2] \) and \( S_{t_j} \in [S_{\text{min}}, S_{\text{max}}] \).

However, at date \( t_1 = t_i \), we rather need the following values for \( S_{t_j} \in [S_{\text{min}}, S_{\text{max}}] \)

\[
C_{i,j} = \mathbb{E}_Q \left[ e^{-r(t_2-t_i)} \left( \frac{S_{t_j} + S_{t_2}}{2} - K \right)^+ | S_{t_1} = S_{t_j} \right] = C_i (S_{t_j}, 2K - S_{t_j})
\]

that is the price of a call on \( S_{t_2} \), with a strike \( 2K - S_{t_j} \) and with the spot price equal to \( S_{t_1} = S_{t_j} \). We therefore use the homogeneity property of the price of the call (12) and we can easily replace the values we have on the grid by the values we need:

\[
C_{i,j} = \frac{2K - S_{t_j}}{K} C_i \left( \frac{KS_{t_j}}{2K - S_{t_j}}, K \right)
\]

Depending on \( j \), there will be different ways to compute these values.

- first case: the call option is already in the money and will be exercised. Its value will be its intrinsic value. If \( S_{t_j} > 2K \), the call option will be exercised for any value of \( S_{t_2} \). Therefore, we have

\[
C_{i,j} = \frac{S_{t_2} e^{-r(t_2-t_1)} + S_{t_2} - K e^{-r(t_2-t_1)}}{2}
\]

- second case: interpolation using the homogeneity property. If \( \frac{KS_{t_j}}{2K - S_{t_j}} \notin [S_{\text{min}}, S_{\text{max}}] \) we will interpolate using the values we already have on the grid. We have implemented a simple linear interpolation.

- third case: extrapolation using the homogeneity property. If \( \frac{KS_{t_j}}{2K - S_{t_j}} \notin [S_{\text{min}}, S_{\text{max}}] \), we have to extrapolate outside the range of values already computed. For this kind of very in-the-money or very out-of-the-money option prices, we can assume that

\[
C_{i,j} = \begin{cases} 
0 & \text{for } \frac{KS_{t_j}}{2K - S_{t_j}} < S_{\text{min}} \\
\frac{2K - S_{t_j}}{K} \left( \frac{KS_{t_j}}{2K - S_{t_j}} - K e^{-r(t_2-t_1)} \right) & \text{for } \frac{KS_{t_j}}{2K - S_{t_j}} > S_{\text{max}}
\end{cases}
\]
3.3 Numerical Results

This methodology gives very satisfactory results. Exhibits 2 and 3 compare the price of the Asian call computed using Monte Carlo simulation with $10^6$ paths and a PDE on a 100x100 grid with $S$ ranging +/-3 standard deviations.

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<thead>
<tr>
<th>Strike</th>
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<th>110%</th>
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<td>Pde Time</td>
<td>3s</td>
<td>3s</td>
</tr>
</tbody>
</table>

Exhibit 2: Price of the Asian call with $r = 0.05$, $t_1 = 1$, $t_2 = 2$ and $\sigma = 0.5$. This was done using a PC with an Athlon 500 Mhz processor. $S_{\text{min}} = 3.10 = \ln(100) - 3 \times 0.5$ and $S_{\text{max}} = 6.10 = \ln(100) + 3 \times 0.5$

<table>
<thead>
<tr>
<th>Strike</th>
<th>100%</th>
<th>110%</th>
</tr>
</thead>
<tbody>
<tr>
<td>PDE</td>
<td>12.40</td>
<td>7.87</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>12.40</td>
<td>7.87</td>
</tr>
<tr>
<td>Time</td>
<td>3s</td>
<td>3s</td>
</tr>
</tbody>
</table>

Exhibit 3: Price of the Asian call with $r = 0.05$, $t_1 = 1$, $t_2 = 2$ and $\sigma = 0.2$. $S_{\text{min}} = 4.00 = \ln(100) - 3 \times 0.2$ and $S_{\text{max}} = 5.20 = \ln(100) + 3 \times 0.2$

4 Extension to the non-homogeneous case

In this section, we use an approximate procedure that would allow the non-homogeneous case to be handle within the framework of section 3, without having to solve the higher order partial differential equation (2.7).

4.1 Taking account for the Smile

4.1.1 The different methods for the smile

The volatility smile is a key concept in option pricing. Research have concentrated over the last ten years extensively on this subject leading to a huge
literature. Traditionally, it is divided into two different approaches: parametric and non parametric ones.

In the first type of methods, the equation of the evolution of the underlying process is specified. This description can consist either in a continuous diffusion process with a so called locally deterministic volatility and implicit lattice methods (Rubinstein (1994), Dupire (1993b) and Derman and Kani (1994) or a continuous diffusion with a stochastic volatility process (Hull and White (1987), Wiggings (1987), Melino and Turnbull (1990), Stein and Stein (1991), Amin and Ng (1993) and Heston (1992)) or a model with jumps (Aase (1993), Ahn and Thompson (1988), Amin (1993), Bates (1991), Jarrow (1984), Merton (1976)).

Other works, close in the spirit, assume constant elasticity of volatility distribution often called power-law (Rubinstein (1994), Cox Ross (1976)). This has also been reformulated by means of a mapping principle between normal and lognormal distributions (Hagan (1998), Pradier and Lewicki (1999)).

The second type of methods is the inference of the underlying distribution from market data, with no assumption on the evolution of the underlying process. This has been called the expansion method. One infers the different terms of the expansion and can rebuild the distribution (Jarrow and Rud (1982), Bouchaud et al. (1998), Abken et al. (1996)).

4.1.2 Case of deterministic volatility

The deterministic volatility model consistent with the smile has been introduced by Dupire (1993a), (1993b), Rubinstein (1994), Derman and Kani (1994). It assumes the volatility structure be a function of the time and the underlying process. The interest of this method lies in its little assumption about the underlying evolution. The ”local volatility” (as opposed to the implied Black Scholes volatility) is proved to be only determined by market data as long as there are enough distinct call options quoted. This implies a liquid market with various call options. It is unfortunately not often the case; and an interpolation procedure is required. However, the main drawback is the instability of the local volatility surface over time and especially for long maturities, for which the inferred structure is frequently not very realistic.

With a known local volatility surface, we can derive the option price as being solution of the modified Black Scholes equation:

\[ C_t + r_t S_t C_s + \frac{1}{2} \sigma^2 (t, S_t) S_t^2 C_{ss} = rC \]
or using the change of variable $X = \log (S)$, we get to

$$
C_t + \left( r_t - \sigma^2 (t, x) \right) C_x + \frac{1}{2} \sigma^2 (t, x) C_{xx} = rC
$$

The Crank Nicolson method leads to the following discretisation scheme

$$
a_{i,j+1} C_{i,j+1} + a_{i,j} C_{i,j} + a_{i,j-1} C_{i,j-1} = a_{i+1,j+1} C_{i+1,j+1} + a_{i+1,j} C_{i+1,j} + a_{i+1,j-1} C_{i+1,j-1}
$$

with

$$
a_{i+1,j+1} = \left( r - \frac{\sigma(t_{i+1},S_i)^2}{2} \right) \frac{1}{4\Delta S} + \frac{\sigma(t_{i+1},S_i)^2}{2} \frac{1}{2\Delta S^2}
$$
$$
a_{i+1,j-1} = \left( r - \frac{\sigma(t_{i+1},S_i)^2}{2} \right) \frac{1}{4\Delta S} + \frac{\sigma(t_{i+1},S_i)^2}{2} \frac{1}{2\Delta S^2}
$$
$$
a_{i,j} = \frac{1}{\Delta T} + \frac{\sigma(t_{i},S_i)^2}{2} \frac{2}{2\Delta S^2} + \frac{r}{2}
$$
$$
a_{i+1,j} = \frac{1}{\Delta T} - \frac{\sigma(t_{i+1},S_i)^2}{2} \frac{2}{2\Delta S^2} - \frac{r}{2}
$$
$$
a_{i,j+1} = \left( r - \frac{\sigma(t_{i},S_i)^2}{2} \right) \frac{1}{4\Delta S} - \frac{\sigma(t_{i},S_i)^2}{2} \frac{1}{2\Delta S^2}
$$
$$
a_{i,j-1} = \left( r - \frac{\sigma(t_{i},S_i)^2}{2} \right) \frac{1}{4\Delta S} + \frac{\sigma(t_{i},S_i)^2}{2} \frac{2}{2\Delta S^2}
$$

4.1.3 Non-homogeneity with a parabolic parameterisation of volatility

A volatility parameterisation for implied volatility $\Sigma^{BS}(K, T)$ (that leads to a certain type of local volatility) widely used on the market is a parabolic dependence with respect to the strike. We have used the following model for the Black Scholes implied volatility in our computations

$$
\Sigma^{BS}(K, T) = \Sigma^{BS}_0 \left( 1 + Sm \frac{K - F(T)}{F(T)} + Cu \left( \frac{K - F(T)}{F(T)} \right)^2 \right)
$$

where $F(T)$ is the forward of maturity $T$, $Sm$ and $Cu$ are some parameters. Usually $Sm < 0$ and $Cu > 0$. We have used a parameterisation for implied volatility since it is straightforward to calibrate the volatility smile to market data.

We have to recall that not all values of $Sm$ and $Cu$ are admissible over a given range for $K$. Indeed, with such a parameterisation, the price of the call may not be convex in $K$ and this allows butterfly arbitrage opportunities (see Hull (1997) for a description of butterfly strategies). We suppose that $Sm$ and $Cu$
Fig. 1. Non-homogeneous case: Prices of a call with a strike equal to 110% of spot. The parameters are $r = 0.05$, $T = 1$, $\Sigma^{BS} = 0.5$, $Sm = -0.25$, $Cu = 0.015$.

are such that the call prices are convex for the $(K,T)$ values we use in the PDE algorithm.

Within this model, the volatility for a given absolute value of the strike depends on the value of $S_0$. The law of $S_t$ depends not only on $\frac{S_t}{S_0}$ but also on $S_0$, and the call prices are no longer homogeneous of degree 1 with respect to $(S_0, K)$.

As shown on Figure 1, acting like in the homogeneous case would induce errors. We propose to correct these errors using the vega of the call price, with the following formula

$$C(\lambda x, \lambda K, \Sigma^{BS}(\lambda K)) \approx \lambda \left[ C(x, K, \Sigma^{BS}(K)) + Vega^{BS}(x, K, \Sigma^{BS}(K)) \left( \Sigma^{BS}(\lambda K) - \Sigma^{BS}(K) \right) \right]$$

This formula is derived from a first order Taylor expansion on the volatility. Because of the dependence of the volatility in the strike, the standard homogeneity property is only true to the first order. The vega term corrects for the non linearity due to the smile. We can neglect terms of higher moments since the difference of implied volatility is of small order and causes the decline of terms of higher orders. This is true because higher order derivatives of the price with respect to the volatility are as well non exploding so that the terms of the Taylor expansion of the type $\frac{\partial^n}{\partial \Sigma^n} C(\lambda x, \lambda K, \Sigma^{BS}(\lambda K))$.
Fig. 2. Vega Correction: Prices of a call with a strike equal to 180% of spot. The parameters are $r = 0.05$, $T = 1$, $\Sigma_0^{BS} = 0.5$, $Sm = -0.25$, $Cu = 0.015$.

\[
\left( \Sigma^{BS}(\lambda K) - \Sigma^{BS}(K) \right)^n / n! \text{ can be neglected for } n \geq 2.
\]

This correction proves to be satisfactory for in-the-money and not very out-of-the-money calls. As shown on Figure 2, it is still acceptable for very out-of-the-money calls. We have to keep in mind that the price of very out-of-the-money calls is small, the error should stay small in absolute value.

We can apply this technique to the interpolation described in Section 3.2. We have to be careful, however, since the Black-Scholes volatility we use in Equation 3 is a forward start volatility, starting on date $t_1$, while Equation 1 gives a volatility valid for options starting at $t = 0$. Computing the forward implied Black-Scholes volatility for all the strikes we need would add a dimension to our PDE algorithm. Instead, we use the smoothened local volatility at time $t_1$. This induces another error, but two facts reduce the impact of this error. First, the smile of market volatility tends to vanish with maturity. Second, if the fixings of the Asian option are close in time, as it is often the case, the local volatility is a good approximation of the forward Black-Scholes volatility between two fixings. Finally, this methodology leads to accurate numerical scheme as shown in the section 5.
4.2 Modelling dividends

Dividend modelling is a complicated issue for equity derivatives pricing. In a model, dividends can be discrete or continuous, proportional or not. It is worthwhile examining for a given problem the implication(s) of assumptions on dividends in terms of realism, simplicity, and efficiency.

4.2.1 Advantage of continuous dividends

Continuous proportional dividends consists in a very tractable solution. Indeed, this assumption changes nothing but the risk free rate, which is diminished by the continuous yield of dividend stream. This hypothesis is often appropriate for an index. The different dividends have different issue dates and smoothen the dividend component. It is not the case for a single stock.

4.2.2 Why using discrete dividends?

For a single stock, it is more appropriate to introduce a non proportional discrete dividend. A more complicated assumption could be as well to have a stochastic dividend. However, we assume that the amount of the dividend is known.

We use the jump condition in our PDE algorithm

\[ C \left( t_i^-, S \right) = C \left( t_i^+, S - D \right) \]

where \( C \left( t, S \right) \) stands for the option price at time \( t \) with and underlying \( S \). \( D \) is the discrete non-proportional dividend.

4.2.3 Effect on homogeneity

Even with a constant volatility, a non proportional dividend prevents us from using the homogeneity property of the Black-Scholes model. The presence of a non proportional dividend shifts the location of the distribution of \( \frac{S_t}{S_0} \) by \( \frac{D}{S_0} \). As in Section 4.1.3, the law of \( S_t \) depends not only on \( \frac{S_t}{S_0} \), but also on \( S_0 \).

As long as the dividend is small compared to the underlying price, we can assume that the effect of the non-proportional dividend on the call price doubles if the spot value of the underlying is divided by two. We therefore propose the following approximation

\[ C \left( \lambda x, \lambda K, D \right) \approx \lambda C \left( x, K, D \right) + (1 - \lambda) \left( C \left( x, K, D \right) - C \left( x, K, 0 \right) \right) \]  (4)
Fig. 3. Dividend Correction: Prices of an at-the-money call. The parameters are $r = 0.05$, $T = 2$, $\sigma = 0.5$, and a dividend of 25 at $t = 1$.

Figure 3 shows that this approximation works well. For important dividend values, however, we have to be careful of negative option prices. A coarse way around this problem is to floor prices to zero. We recall that call prices at these low spot levels are already very small. The absolute error should not be very important.

In our implementation, we run twice the algorithm between two fixings: once with the dividends and once without. We then have all the data we need to use Equation 4.

5 Numerical Results

This section presents numerical results of our methodology in a simple framework. We price and Asian call with two fixings on dates $t_1$ and $t_2$. The continuously compounding risk-free interest rate is supposed to be constant. The ex-dividend date is $\frac{t_1 + t_2}{2}$. Finally, we have modelled the local volatility directly

$$\sigma(t, S) = \sigma_0 \left( 1 + sm \frac{S - S_{ref}}{S_{ref}} + cu \left( \frac{S - S_{ref}}{S_{ref}} \right)^2 \right)$$  \hspace{1cm} (1)$$

with boundaries at 0 and 10. More realistic applications would use an implied local volatility instead, as described in Section 4.1.2.
Our results will be benchmarked with a 2-dimension PDE solver.

5.1 The choice of the finite-differences mesh

Exhibit 4 presents results for different numbers of time steps and space steps. A more complete study should examine the impact of moneyness on the convergence.

<table>
<thead>
<tr>
<th>time and space steps</th>
<th>20</th>
<th>50</th>
<th>100</th>
<th>200</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 standard deviation</td>
<td>24.18</td>
<td>24.15</td>
<td>24.13</td>
<td>24.11</td>
</tr>
<tr>
<td>3 standard deviations</td>
<td>24.32</td>
<td>24.44</td>
<td>24.46</td>
<td>24.47</td>
</tr>
<tr>
<td>6 standard deviations</td>
<td>23.67</td>
<td>24.37</td>
<td>24.44</td>
<td>24.46</td>
</tr>
<tr>
<td>Time</td>
<td>1s</td>
<td>2s</td>
<td>5s</td>
<td>10s</td>
</tr>
</tbody>
</table>

Exhibit 4: Prices of an at-the-money Asian call with $t_1 = 1$, $t_2 = 2$, $\sigma = 0.5$ and no smile or dividend.

In the remaining calculations, we take a 100x100 mesh and a range of 3 standard deviations for the underlying, which seems to be a good compromise between precision and computing time.

5.2 The "vega correction" in the case of a volatility smile

Figure 4 compares the value profiles we get on our grid at a date just before the fixing, i.e. just after we have used our interpolation procedure. Thanks to the vega correction of Section 4.1.3, the interpolation profile matches quite well the "real" profile obtained through a 2-dimensional PDE.

Consequently, the accuracy of upfront prices is satisfactory, as shown on Exhibit 5
Fig. 4. Smile model: Price of a 110% call just before the fixing. The parameters are $\sigma = 0.5$, $sm = -1$, $cu = 0$ and no dividend.

<table>
<thead>
<tr>
<th>$sm$ parameter</th>
<th>0</th>
<th>-0.2</th>
<th>-0.5</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D benchmark</td>
<td>20.81</td>
<td>20.47</td>
<td>19.92</td>
<td>18.93</td>
</tr>
<tr>
<td>with vega correction</td>
<td>20.81</td>
<td>20.51</td>
<td>19.95</td>
<td>18.91</td>
</tr>
<tr>
<td>without vega correction</td>
<td>20.81</td>
<td>20.50</td>
<td>20.00</td>
<td>19.08</td>
</tr>
</tbody>
</table>

Exhibit 5: Prices of a 110% Asian call with $\sigma_0 = 0.5$, $cu = 0$ and no dividend.

5.3 The "dividend correction" in the case of non-proportional dividends

The dividend correction of Section 4.2.3 gives good results on the fixing date profile (see Figure 5.)

And this transposes into very good results for the upfront price, as shown on Exhibit 6.
Fig. 5. Dividend Case: Price of an at-the-money call just before the fixing. The parameters are $\sigma = 0.5$, no smile and a dividend of 30.

<table>
<thead>
<tr>
<th>dividend value</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>3D benchmark</td>
<td>24.46</td>
<td>23.38</td>
<td>22.25</td>
<td>18.22</td>
</tr>
<tr>
<td>with dividend correction</td>
<td>24.46</td>
<td>23.36</td>
<td>22.26</td>
<td>18.25</td>
</tr>
<tr>
<td>without dividend correction</td>
<td>24.46</td>
<td>23.55</td>
<td>21.77</td>
<td>19.45</td>
</tr>
</tbody>
</table>

**Exhibit 6**: Prices of an at-the-money Asian call with, $\sigma = 0.5$ and no smile. Since we take the same case as in table 4, we have the same price when having no dividend.

6 Conclusion

In this paper, we have seen that we can price an Asian option efficiently with a 1-dimension PDE method. The contribution of this paper lies in two ways. We have examined the particular case of the discrete Asian option which is often ignored in the previous literature. We have used the homogeneity of the Black Scholes underlying to reduce the dimension. We have extended the results of Rogers and Shi (1995) to non-constant volatility structure. We have seen the importance of the homogeneity property. It is only in the case of the Black Scholes diffusion that the problem reduces to a one dimensional one. Indeed, with a deterministic volatility like in the Dupire (1993a), (1993b) and Derman and Kani (1994) models, an other variable needs to be added. This is because we have lost the homogeneity property. However, this homogeneity is coarsely satisfied and can be corrected. This enables us to keep on using the backward
propagation in one dimension as in Black Scholes. We have examined the impact of certain numerical specification for the finite difference method as well as the impact of discrete dividends.

There are many possible extensions to this paper. The first one would consist in finding additional features on the relationship between the different calls for non-constant volatility structure. The homogeneity seems to handle this quite well. However, we have no boundary on the error term. A second enlargement of this work concerns other path dependent options, like ratchet options. The approach adopted here should be adaptable to this kind of options.

References


